Linguistics and Some Aspects of Its Underlying Dynamics

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In recent years, central components of a new approach to linguistics, the Minimalist Program, have come closer to physics. In this paper, an interesting and productive isomorphism is established between minimalist structure, algebraic structures, and many-body field theory opening new avenues of inquiry on the dynamics underlying some central aspects of linguistics. Features such as the unconstrained nature of recursive Merge, the difference between pronounced and un-pronounced copies of elements in a sentence, and the Fibonacci sequence in the syntactic derivation of sentence structures, are shown to be accessible to representation in terms of algebraic formalism.

Keywords: algebraic methods; coherent states; deformed Hopf algebra; Fibonacci progression; self-similarity

1. Introduction

The linguistic component of the present work is based on 'generative grammar' (GG; Chomsky 1955 et seq.). Our work deals with a relatively recent version of the theory called the 'Minimalist Program' (MP; Chomsky 1995) and more particularly with a very recent further development over the past few years that has brought linguistics even closer to physics. We will not go into the debate pro and con GG, embodied in a vast literature, out of which we indicate only some basic references.[1]

We show how some MP features are quite well suited to a mathematical representation in terms of algebraic methods and tools. This goes beyond a pure, although difficult, formal exercise, since it reveals the dynamics underlying aspects of

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[1] A use-based explanation of language is offered in Tomasello (2003) and Bybee (2007); for counters, see Wexler (2002), Crain et al. (2009), and Pietroski (2003). For a statistical approach to syntax and language learning, see Reali & Christiansen (2005), Perfors et al. (2011), and Christiansen & Chater (2015); for counters, see Berwick et al. (2011) and Berwick et al. (2013a).

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the MP, which thus appears much richer than one might had suspected. Especially, it uncovers many contact points of the linguistic structure with concrete properties of nonlinear algebraic formalism commonly used in the description of physical systems. Although in our scheme linguistic structures are classical ones, we find that an isomorphism can be established between the MP linguistic structure and many-body field theory. In our opinion, a very rewarding result, no matter from which standpoint one looks at, e.g. recognizing the deep dynamical processes underlying the MP linguistic structures, or, vice versa, the linguistic content of the many-body formalism. The plan of the paper is the following. In section 2, the most relevant aspects of the MP are presented, including a re-analysis of X-bar trees, their self-similarity properties, and their formalization under our schema. In section 3 and its subsections, the interfaces, the manifold of concepts, and the copies of lexical elements are discussed. Section 4 is devoted to final remarks where comments on the entropy and the arrow of time are presented. Finally, in the Appendices A–C, some details of the mathematical formalism are reported. Some properties of the Fibonacci matrix are discussed in Appendix D.

2. The Relevant Components of the Minimalist Program

In the MP, accrued emphasis is put on "third factors of language design,"[2] that is, principles that are not specific to language, nor specific to biological systems; basically, minimal (strictly local) search, minimal computation. In other words, the physics and the mathematics of language. For a broader approach to language and language evolution, see Perlovsky & Sakai (2014) and Perlovsky (2013) and references therein. The most basic and simplest operation now is binary Merge.

The binary, unordered set created by Merge is then Merged with a third element from the lexicon. This binary Merge is recursively repeated until the whole sentence is terminated. The syntactic process, called 'derivation' is similar to a proof ending when the sentence is terminated. In more complex sentences, with subordinates, relatives, or embeddings, the process goes on until the derivation finally stops (Chomsky 2001).

There are intermediate cyclic points of derivational (computational) closure, called Phases.[3] The syntactic derivation (the specific mental computation) stops when a Phase is reached, and then a higher Phase is opened. The process continues inside-out, building higher and higher components in the syntactic hierarchy. All these recursive operations are binary and leave the items being merged unaltered.

There are few components overall: External Merge, Internal Merge, Agree, and the Labeling Algorithm—to which we will return later. There is nothing else in syntax; it is therefore called Narrow Syntax.

In the previous theory of Government and Binding (Chomsky 1981; Haegeman 1991), there where more components and entities. These are now, in Minimalism, subsumed under more basic operations, under a constraint of strict locality. The head gives the name to the constituent it generates (nouns to Noun Phrases, verbs

[2] The other two factors are: genetic predispositions and peculiarities of the local language that the child has to learn (Chomsky 2005).

[3] For greater clarity, we will use upper case P for Phases in syntax and lower case p for phases in physics.

to Verb Phrases, and so on). More generally, we have an {H, XP} construction, a Head and a Phrase. The X in XP is a generalization, meaning that it can be any one of a great variety of phrasal categories.

What were previously (in the theory of Government and Binding) called 'empty categories' (because they are not pronounced or written) are now simplified in terms of copies. Copies come for free, so to speak, because they are elements already present in previous steps of the derivation, for instance, items extracted from the lexicon.[4] The replacement of empty categories with copies of lexical elements (pronounced or un-pronounced—a distinction to which we return below) is a step in simplification and has proved to be a legitimate move in many cases.

In GG the condition of 'strict locality' applies to the structure of the sentence, not necessarily to what is, or is not, 'close' on the surface of the sentence. It has been emphasized that one cannot just count the number of words separating the affected elements in the sentence. What counts are the number and kind of nodes separating the affected elements in the syntactic tree. In GG, long before the MP, this central property of syntax had been called 'structure dependence'. It constitutes a sharp departure from many old and new anti-generativist approaches to language based on statistics or conventions of use.

The generative theory of grammar has allowed a deep analysis of many languages and dialects. It also turned out that the vast majority of all Phrases had the same structure, X-bar structure, which is recursive: An element of the structure (a node of the X-bar tree) can contain another X-bar structure, and so on; recursively, indefinitely.[5]

Perhaps, this is a good point where to insert in our presentation a first part of our algebraic formalization. In fact, we will see that we obtain in a straightforward way the recursivity, or self-similarity, of the X-bar structures.

2.1. *X-Bar Structures, Their Self-Similarity, and the Breakdown of Time Reversal Symmetry*

It has become standard in GG to construct syntactic trees that have only two branches departing from each node. This is referred to as 'binary branching' (Kayne 1984). In fact, we have a collection of binary entities. Lexical items are represented, by useful convention, as $(+,\ -)$, which is written in the matrix formalism by using the standard vector notation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for '$+$' and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for '$-$'. The notation for $(+a, -b)$ is then $a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. Thus, Nouns are $(+N, -V)$, Verbs as $(+V, -N)$. This notation can be usefully extended to Phrasal Heads $(+H, -C)$ and Complements $(+C, -H)$. In the syntactic derivation, we have Terminal nodes $(+T)$ and nonterminal nodes $(-T)$. Copies of lexical items, or of larger structures, in a sentence can be pronounced $(+Pr)$ or not-pronounced $(-Pr)$. Recursive applications of Merge may produce a Phase $(+Ph)$ or not $(-Ph)$. The most basic

4 Different languages treat the copies differently. In most languages only the higher copy is pronounced, but there are languages in which the lower copy is pronounced and also languages in which all copies are pronounced. In the latter case, this applies to 'short' elements (equivalent to the English 'who', 'which', and similar), never to whole Noun Phrases.

5 The X is a portmanteau symbol, covering most kinds of Phrases.

syntactic operation, Merge, generates a binary set. This suggests to us to formalize the binary branching in terms of standard formalism of vector or state spaces and matrix multiplications. In the following we will also use the shorthand notation $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In general, we may consider a collection of N objects ('particles' or 'lexical elements'), which in a standard fashion can be labeled by $i = 1, 2, \dots N$ as $|0\rangle_i$ and $|1\rangle_i$.

In Appendix A we introduce so-called Pauli matrices and the matrices σ^+ and σ^-. The interest in the matrices $\sigma^\pm$ is due to the fact that they generate the transitions between the two states $|0\rangle$ and $|1\rangle$:

(1) $$\sigma^-|1\rangle = |0\rangle \ , \quad \sigma^+|0\rangle = |1\rangle \ , \quad \sigma^-|0\rangle = 0 \ , \quad \sigma^+|1\rangle = 0$$

In order to see how 'binary Merge' between two states is generated, consider these two states $|0\rangle$ and $|1\rangle$. They may represent two lexical elements or two levels of the same lexical element. In the following we will consider generalization to the collection of N elements and restore the index i, which now for simplicity we omit.

In physics, $|1\rangle$ is said to be the excited state with respect to $|0\rangle$ which is called the 'vacuum' or the ground state. The process leading from $|0\rangle$ to $|1\rangle$ is called the excitation process and the one leading from $|1\rangle$ to $|0\rangle$ is called the decay process of the $|1\rangle$ state. We thus start with $|0\rangle$. Of course, we want to move on from the state $|0\rangle$. Here and in the following we do not consider the (trivial) possibility to remain in the initial state $|0\rangle$, which is equivalent to "nothing happens."[6] The interesting possibility is the one offered by the process leading from $|0\rangle$ to $|1\rangle$. According to (1), this process is obtained by applying σ^+ to $|0\rangle$:

(2) $$|0\rangle \quad \rightarrow \quad \sigma^+|0\rangle = |1\rangle$$

Thus, as a first single step the state $|1\rangle$ has been singled out. By 'single step' we mean that we have multiplied $|1\rangle$ by one single matrix, the σ^+, not by a product of σ's. In this connection, consider that $\sigma^+ \sigma^+ = 0 = \sigma^- \sigma^-$. Therefore, the only possibilities to step forward of a single step is given in (2), and from there, one more single step is obtained as:

(3) $$\nearrow \ \sigma^-|1\rangle = |0\rangle$$
$$|0\rangle \quad \rightarrow \quad \sigma^+|0\rangle = |1\rangle \quad \rightarrow$$
(4) $$\searrow \ \sigma^+\sigma^-|1\rangle = |1\rangle$$

Note that application of $\sigma^+\sigma^-$ is considered to produce a single step, since it is equivalent to the application of the unit matrix I to $|1\rangle$. In general, for any integer n, $(\sigma^+\sigma^-)^n\,|1\rangle = 1 \times |1\rangle$. Note also that (3) describes the decay process of the excited state $|1\rangle$ to $|0\rangle$. The equation (4) describes the 'persistence' in the excited

[6] The physical meaning of this is that we neglect fluctuations in the ground state, which can be described by $\sigma^- \sigma^+|0\rangle = 1|0\rangle$, i.e. $|0\rangle \rightarrow |0\rangle$. In the quantum formalism, this is achieved by considering the so-called 'normal ordering' or 'Wick product' of the operators. But here we do not need to insist further on such an issue.

state, which represents a dynamically non-trivial possibility and thus we have to consider it. One more step forward leads us to (5)–(7) and so on:

(5) $$\nearrow \ \sigma^-|1\rangle = |0\rangle \ \rightarrow \ \sigma^+|0\rangle \ = \ |1\rangle$$

$$|0\rangle \ \rightarrow \ \sigma^+|0\rangle = |1\rangle \ \rightarrow$$

(6) $$\searrow \ \sigma^+\sigma^-|1\rangle = |1\rangle \ \begin{matrix} \nearrow \sigma^-|1\rangle = |0\rangle \\ \searrow \sigma^+\sigma^-|1\rangle = |1\rangle \end{matrix}$$

(7) 1 1 2 3

At each step, new branching points $\begin{matrix}\nearrow\\ \searrow\end{matrix}$ (new nodes of the X-bar tree) are obtained and the X-bar tree is generated by recursive σ 'operations', i.e. by multiplying $|0\rangle$ and $|1\rangle$ by the σ matrices, which we also call σ 'operators'. The set of these operations constitute what is named, in technical terms, the "SU(2) transformation group" (Perelomov 1986; see also Appendix A).

The conclusion at this point is that we have the 'number of the states' in these first steps in the sequence: $1 \ 1 \ 2 \ 3$, starting with $|0\rangle$, [one state] , then in equations (2) [one state], (3) and (4) [2 states], and (5) and (6) [3 states], respectively, (cf. (7)).

From here, from the two $|1\rangle$'s, we will have in the next step two $|0\rangle$'s and two $|1\rangle$'s, and from the $|0\rangle$ we will get one single $|1\rangle$—in total 5 states: $1 \ 1 \ 2 \ 3 \ 5$. We will get thus, in the subsequent steps, other states, and their numbers obtained at each step are in the Fibonacci progression $(\{F_n\}, F_0 \equiv 0)$ with the ones obtained in previous steps. In general, suppose that at the step F_{p+q}, one has p states $|0\rangle$ and q states $|1\rangle$; in the next step we will have: $(p+q)\,|1\rangle$ and $q\,|0\rangle$, $F_{q+(p+q)}$. In the subsequent step: $(p+2q)\,|1\rangle$ and $(p+q)\,|0\rangle$, a total of states $2p+3q = (q+p+q) + (p+q)$, i.e. the sum of the states in the previous two steps, which agrees with the rule of the Fibonacci progression construction.[7]

It is interesting to remark that $(\sigma^+\sigma^-)^n|1\rangle = 1 \times |1\rangle$, for any integer n, can be thought of as a 'fluctuating' process: The σ^- brings $|1\rangle$ down to $|0\rangle$, and σ^+ again up to $|1\rangle$, and so on for any integer n: $\sigma^+\sigma^-$ induces fluctuations $|1\rangle \ \leftrightarrows \ |0\rangle \ \leftrightarrows \ |1\rangle$ (through the 'virtual' state $|0\rangle$); this is the meaning of the fact above observed that $\sigma^+\sigma^-$ is equivalent to 1 at any integer power n when operating on $|1\rangle$. This 'fluctuating activity' corresponds, in the syntactic derivation, to successive applications of Merge. Simplifying a bit, when recursive Merge reaches the topmost node of a Phase, that is, a point of computational closure, everything underneath, in the tree, becomes off limit. The condition called 'Phase Impenetrability Condition' (PIC) (Chomsky 2000, 2001; Richards 2007; Gallego 2012) specifies that nothing in a lower Phase is accessible to the syntactic operations that create the immediately higher Phase. The syntactic objects of the lower Phase and the lower Phase itself are dynamically 'demoted' to a $|0\rangle$ state. The 'fluctuating activity' is also much suggestive when one thinks of the processes (of milliseconds or so) in the selection of lexical items and the recursive Merge of these into syntactic objects.

[7] There are many ways to capture how natural phenomena generate the Fibonacci, or F progression, both in inorganic and organic systems (especially in botanic structures). The present approach is, we think, particularly elegant and especially close to how the F progression is generated in syntactic structures.

Summarizing, we have described the 'action' on the state $|0\rangle$ and $|1\rangle$ by application (multiplication) of the sigma matrices. In the physics jargon, one says that the 'dynamics' of a system is defined once the rule of 'how to go' from one step to the next one in the system evolution is found. Accordingly, in the present case, we can say that the X-bar tree (or F tree) has been obtained as a result of the $SU(2)$ dynamics (namely the set of operations induced by products of σ matrices), with the additional result that its multiplicity of states, its recursivity or self-similarity properties turn out to be described by the Fibonacci progression.

We also observe that the full set of σ^+ and σ^- products compatible with the $SU(2)$ algebra (the products used above and leading, as we have seen, to the F progression) generates what is called the Jaynes-Cummings-like dynamics, which has a wide range of physical applications (see e.g. Gerry & Knight 2005; Blasone et al. 2011). Thus our construction presents features which certainly deserve much attention, since we now have that the X-bar tree, which plays so a crucial role in the MP, arises as a result of a *dynamical* model in linguistic, its recursive property being related to the self-similarity property of the Fibonacci progression. The paramount importance of the Fibonacci progression in language has been stressed by Medeiros (2008), Idsardi & Uriagereka (2009), Piattelli-Palmarini & Uriagereka (2004, 2008), and in Medeiros & Piattelli-Palmarini (in press). References therein cover a variety of instantiations of Fibonacci structures in natural systems ranging from binary stars to ferromagnetic droplets, from botanic forms to brain waves and beyond.

We close this subsection by observing that at any given step of the X-bar tree (the F tree), the simple knowledge of the state $|0\rangle$ or $|1\rangle$ is not sufficient in order to know its parent state in the previous step; we should also know which one is the branch we are on. This in part corresponds to the PIC mentioned above and to one of the major problems in all of contemporary linguistic theory. In speaking and reading we proceed left to right, from the 'outside' (the main sentence), to the 'inside' (subordinate sentence), but the syntactic derivation proceeds from right to left, from inside out. This creates a conflict, namely that presumably the construction of Phases—that is, of periodic points of closure—solves (Piattelli-Palmarini & Uriagereka 2004, 2005, 2008).

While the tree construction (the 'way forwards') is fully determined by the σ's operations, the 'way backwards', as said, is not uniquely determined solely by the knowledge of the state $|0\rangle$ or $|1\rangle$. On the other hand, suppose one goes backwards of, say, q steps starting from a given, say, $|1\rangle$ (or $|0\rangle$). Then returning to such a specific state is no more guaranteed since at each branching point one has to chose which way to go (unless one keeps memory of its previous path, the Ariadne's thread…). In the syntactic derivation, 'forward' consists in building further structure from the inside out, from right to left, proceeding upwards in the syntactic tree. The opposite, 'backwards', consists in the derivation 'looking down' to lower levels. The PIC, as we have just seen, constrains this operation to a strict minimum. Omitting details, only the leftmost (and topmost) 'edge' of the lower Phase is (quite briefly) still accessible to the operations building the next higher Phase.

The lesson is that, parameterizing by time the moving over the X-bar tree, time-reversal symmetry is broken. In other words, as seen above, the 'way forwards' and the 'way backwards' cannot be trivially exchanged, which means that on the axis of the coordinate representing the time (the time axis), the origin—say the time t_0—

is not a symmetric point under exchange of the forward and backward direction, indeed, which in turn forbids that one can choose it or move it on the time axis arbitrarily. In such a case, according to the Noether theorem, the system energy is not conserved. The system may exchange, release or receive, energy with its environment. It is an open or dissipative system. We therefore need to deal with the formalism specially devised for dissipative systems. We will consider such a problem in the following. Before that we need to comment briefly in the following section on the 'interfaces', namely the conceptual intentional (semantic) system (CI) and the sensory-motor system (SM), to which Narrow Syntax has to make contact.

3. The Interfaces

Narrow Syntax has to make contact (has to interface) with two distinct systems: the conceptual intentional (semantic) system (CI) and the sensory-motor (articulation, auditory, or visual perception) system (SM). Language, for centuries, has been correctly conceived as sounds with meanings.[8] But it is better now conceptualized as meanings with sounds, because Narrow Syntax is optimized to interface with the CI system, not so much with the SM system. CI 'sees' all copies, and interprets them, but at the SM interface only one copy is pronounced (usually the higher copy; see footnote 4), while the other copy (or copies) remain(s) silent (deleted at SM):

(8) Which books did you read [books]?

The rightmost (hierarchically lower) copy in English and in many other languages is not pronounced. We see that 'copies' now become important objects in the linguistic structure. We will show how this can be accounted for in our modeling. Until 2012, the 'optimality' of Narrow Syntax with regard to the CI system was supposed to operate as follows: There are features that are 'meaningful', called interpretable features, which CI can understand, and other features that are uninterpretable, meaningless. From 2012 on, the bold hypothesis is that Merge does not form sets that have a category, not any more. It works freely and without constraints (a bit like Feynman's sum of all histories, before amplitudes give the wave function). It is 'only' at the interface with CI that categories are needed (CI needs labeled heads: which one is a verb, which one a noun, an adjective etc.).[9] A minimal search process called the Labeling Algorithm is what does this job (Chomsky 2013, in press). In this framework, categorization and non-commutativity are only necessary at the CI interface. Order is important, obviously, at the SM interface (what to pronounce first, second etc., and what not to pronounce at all—deleted copies), but there is strong evidence that order does not appear at the CI interface. Order

[8] Sound is the traditional expression, but we now know that it is unduly too restrictive: This should extend to gestures in sign languages (see the classic analysis of American Sign Language by Klima & Bellugi 1979 and many studies ever since) to touch in deaf-and-blind subjects (C. Chomsky 1969, 1986)

[9] It needs more than this: If XP is a VP at CI (the highest node, a Complementizer Phrase), then the mapping from Narrow Syntax to the SM system (externalization) must also know that it is a VP . Therefore labeling must be done at Transfer, so that the information goes to both interfaces.

is probably a reflex of the SM system, not feeding Narrow Syntax or CI.[10] And categorization has to be the same at CI for interpretation at SM and for externalization. Today, some syntacticians try to shoehorn the previous analysis into this more stringent picture. Not everyone is persuaded that it can be done completely. But interesting explanations with elegant simplifications have been obtained already (see, among many, Berwick et al. 2013b; Cecchetto & Donati 2010; van Gelderen 2014; Hornstein 1999; Hornstein et al. 2005). In essence: Explain and unify in terms of unconstrained Merge and the Labeling Algorithm many (ideally, all the) special properties of syntax. In many linguistic expressions, nothing is invoked beyond the simplest computational operation Merge and reasonable interpretations of general principles of strict locality and Minimal Computation (MC). It's third factors (physics) all the way.

3.1. The Manifold of Concepts

We are now ready to resume the discussion of the algebraic formalism. Our first task is to consider the whole set of N elements introduced in the subsection 2.1 and thus restore the subscript i labeling each element in the set of N elements.

One may regard the collection of the associated states as the one at a given step of high multiplicity in the Fibonacci tree. Since N can be as large as one wants, we may always have a state which is the direct product of a large number (in principle, an infinite number, hence one needs field theories) of factor states, $\Pi_{i=1,N}|s_i\rangle \equiv |s_1\rangle \otimes |s_2\rangle \otimes \ldots |s_i\rangle \otimes \ldots \equiv |s_1, s_2, \ldots s_i \ldots\rangle$, with $s_i = 0$ or 1 for each $i = 1, 2, \ldots, N$. The most general state. denote it by $|l\rangle$, is then a superposition of all states with l elements in $|1\rangle$ and $N - l$ elements in $|0\rangle$. Its explicit form is given in Appendix B. The difference between the number of elements in $|1\rangle$ and the one of the elements in $|0\rangle$ is measured by σ_3 and is given by $\langle l|\sigma_3|l\rangle = l - \frac{1}{2}N$. This quantity is called the order parameter. Its being non-zero signals that the $SU(2)$ symmetry is broken.[11]

In Appendix B (see also Beige et al. 2005; De Concini & Vitiello 1976), it is shown that in the large N limit the su(2) algebra of the σ matrices, represented in the space of the $|l\rangle$ states, for any l, and written in terms of $S^{\pm}$ and $S_3 \equiv \sigma_3$, where $S^{\pm} = \sigma^{\pm}/\sqrt{N}$, transforms (rearranges) into the algebra in (9) .

(9) $$[S_3, S^{\pm}] = \pm S^{\pm}\,, \quad [S^-, S^+] = 1$$

The result (9) is a central result. Its physical meaning is that, as a consequence of the spontaneous breakdown of symmetry, long range correlation modes (the Nambu-Goldstone modes) are *dynamically* generated (the Goldstone theorem; Goldstone et al. 1962). These Nambu-Goldstone modes represent collective waves spanning the whole system and are here represented by the ladder $S^{\pm}$ operators. They are the carrier of the *ordering* information through the system volume (Shah et al. 1974; De Concini & Vitiello 1976; Umezawa 1993; Blasone et al. 2011).

10 There are two notions of order to be taken into account: ordering of the syntactic operations and ordering of the items in the externalized linguistic expression. Here we have dealt with the latter, while a treatment of the first comes in what follows.

11 The phenomenon of spontaneous symmetry breakdown is throughly studied in many-body physics. For example, in the case of the electrical or magnetic dipoles, the order parameter provides the measure of the polarization or magnetization, respectively.

Order thus appears as a *collective* dynamical property of the system. The order parameter provides indeed a measure of the system ordering. Different degrees of ordering correspond to different values, in a continuous range of variability, of the order parameter, thus denoting different, i.e. physically inequivalent *phases* of the system.

When spontaneous symmetry occurs, the system may be found therefore in different dynamical regimes or physical phases. These are described by different spaces of the states of the system each one labeled by a specific value assumed by the order parameter. Such a process of dynamical generation of physically different phases, each one characterized by collective, coherent waves, represented by the ladder operators $S^{\pm}$, is called *foliation* in the jargon of quantum field theory (Celeghini et al. 1992; Vitiello 1995; see also subsection 3.2 and Appendix C).

In GG, the phenomenon of symmetry breaking—the anti-symmetry of syntax and the dynamic anti-symmetry of syntax—have been cogently argued for by Kayne (1994) and Moro (2000), respectively, for example. This is in part why issues about the status of X-bar (as part of Narrow Syntax or as an emergent configuration of recursive binary Merge) have been recently debated (Chomsky 2013, in press; see also Medeiros & Piattelli-Palmarini, in press). In essence, if Merge is unconstrained and does not, in itself, produce ordered sets, we have an initial symmetry (i.e. before the interfaces with CI and SM). Labeling and ordering at the interfaces break this symmetry and create order. This process does not involve any material transfer, something that is obviously excluded in the case of language.[12]

We thus realize that, due to the spontaneous symmetry breakdown, our system has undergone a formidable dynamical transition, moving from the regime of being a collection of elementary components (lexical elements) to the regime of collective, coherent $S^{\pm}$ fields. Our main assumption at this point is to identify a specific *conceptual, meaningful* linguistic content (a Logical Form, LF)[13] with the collective coherent phase associated to a specific value of the order parameter. The semantic level, characterized by a *continuum* of concepts or meanings (the 'manifold of concepts'), thus emerges as a dynamical process out of the syntactic background of lexical elements, in a way much similar (mathematically isomorph) to the one by which macroscopic system properties emerge as a coherent physical phase out of a collection of elementary components at a microscopic (atomistic) level in many-body physics (Umezawa 1993; Blasone et al. 2011).

In conclusion, we can now give a quantitative characterization of the 'interfaces' where the Narrow Syntax has to make contact with the CI system: When interfaces are met we have the spontaneous breakdown of symmetry in the large N limit. It is there that a specific meaning or 'concept' arises from a 'continuous' context of possible concepts by selecting out one representation of the algebra from many of them

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12 A comparison due to the Oxford particle physicist Frank Close is the following (Close 2011): Imagine several guests sitting at a very large circular dinner table. Each has a napkin on his or her right and one on the left. They are uncertain about which one to pick up. Until a more daring guest decides to pick up the one on (say) the right. Everyone else follows and we have a 'wave' of napkin pickups. The underlying symmetry is broken. No movement of matter, no forces applied. In analogous situations in physics, a Nambu-Goldstone boson (a mass-less particle) is thus generated.

13 The notion of LF as the last syntactic input to full meaning is well consolidated in GG and has been since the pioneering work of Higginbotham & May (1981) and May (1985).

'unitarily inequivalent' among themselves (each corresponding to a different concept) (Vitiello 1995). The concept appears at that point as a collective mode, not a result of associative process pulling together bits and little lexical pieces, words etc. The collectiveness comes from the 'phase coherence', whose carriers are the collective Nambu-Goldstone $S^{\pm}$ fields. We also understand why "only at the interfaces the issues of ordering become relevant" (cf. previous subsection). Order indeed is lack of symmetry and it can only appear when this is spontaneously broken.

For the same reason, categorization and non-commutativity (and order) are only necessary at the CI interface. Indeed, only at the large N limit CI needs labeled heads: which one is a verb, which one a noun, an adjective etc. We have seen that the formal construction of the binary Merge does not require labeled structures (Noun, Verb, Adjective, Preposition etc.). The necessity of labeling (through the Labeling Algorithm) only arises at the interface with meaning. Interpreting the different constituents (Phrases) is a necessity for the CI system, with the formal label of a syntactic object triggering different intentional landscapes. Once the Narrow Syntax has made contact with the CI system, through the action-perception cycle (Vitiello 1995) of the cortex dynamics, the SM system gets also involved and therefore the linguistic structures can be externalized, allowing to communicate to other speakers all the required subtleties of meaning.

The formalism here presented thus endorses Chomsky's thesis that Merge is unconstrained, and that issues of labeling (headedness, categorization of lexical items) and ordering only arise at the interfaces of Narrow Syntax with the CI and the SM systems.

3.2. Copies of Lexical Elements

We now consider the feature of the copies of lexical elements in the MP. At the end of subsection 2.1, we have observed that time-reversal symmetry is broken moving along the X-bar tree. We saw that when the breakdown of time-reversal symmetry occurs, one cannot treat the system as a closed system. It is a dissipative system and from the standpoint of the algebraic formalism, this means that one has to set up a proper mathematical scheme, which is achieved by doubling the system degrees of freedom (Celeghini et al. 1992). This goes as follows.

Consider a dissipative system, say A. It is an open system interacting with the environment in which it is embedded, denote it with $\tilde{A}$. In order to carry on the analysis of the system properties one cannot avoid to consider the fluxes of energy, matter, information, etc. exchanged between the system A and its environment $\tilde{A}$. This implies that the study of the dissipative system cannot ignore the study also of the properties and features of the environment. Thus one needs to consider both, the system and its environment. This means that, instead of considering the system A separated from the environment, one is brought to consider algebraic forms including both of them, $\{A, \tilde{A}\}$, namely $A \to \{A, \tilde{A}\}$. However, one must pay attention in treating the system elements and the environment element, since in general, the system elements cannot be exchanged or confused with the elements of the bath or environment in which the system is embedded. They need to be considered on a different footing. This is obtained by introducing a 'weight factor', or 'deformation parameter', say θ, with different values for A and $\tilde{A}$ (Celeghini et al. 1998; Blasone et al. 2011). Such a procedure may be formulated in a precise manner and goes under the name of 'deformed Hopf algebra structure', which is a noncommutative algebra. See Appendix C for introductory details. The conclusion is that one has now to deal with a 'doubled' system: A and its double or 'copy' $\tilde{A}$. As a matter of fact, since the fluxes between A and $\tilde{A}$ must be balanced, one may think indeed of $\tilde{A}$ as a 'copy' of A, in the sense that $\tilde{A}$ represents the sink where, say, the energy from the source A goes, and vice versa, A also represents the sink where the energy from the source $\tilde{A}$ goes. The 'tilde' operators $\tilde{A}$ thus denote the doubled operators in the doubling of the algebra $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ (see also Appendix C).

Note that, when considering the elements of A (and $\tilde{A}$), one should use subscripts, say $\mathbf{k}$, denoting characterizing properties of the A (and $\tilde{A}$) modes, e.g. $A_{\mathbf{k}}$. For simplicity we omit such subscripts as far as no misunderstanding occurs.

Simplifying a bit, the doubling of the space and of the operators creates a strict correspondence between each operator and its 'double' (the tilde operator). This two-way interaction is quite specific. In the case of language, each copy interacts with the initial (in a sense, the 'original') element and meaning is accordingly extracted at CI. As CI well 'understands', the interpretation is determined by this dual correspondence.

Denote now by $|0\rangle \equiv |0\rangle \times |0\rangle$ the state annihilated by A and $\tilde{A}$: $A|0\rangle = 0 = \tilde{A}|0\rangle$ (the vacuum state). By proper algebraic operations (see Blasone et al. (2011) and Celeghini et al. (1998) for the technical details) one may show that starting from the operators A and $\tilde{A}$, the operators $A(\theta)$ and $\tilde{A}(\theta)$ may be obtained, such that they do not annihilate $|0\rangle$. Let us denote the state annihilated by these operators by $|0(\theta)\rangle_{\mathcal{N}}$. Its explicit form is given in Appendix C.

The vacuum state $|0(\theta)\rangle_{\mathcal{N}}$ is a well normalized state: ${}_{\mathcal{N}}\langle 0(\theta)|0(\theta)\rangle_{\mathcal{N}} = 1$. The meaning of the subscript $\mathcal{N}$ is clarified below (see the comments after equation (10)). We remark that the vacuum state $|0(\theta)\rangle_{\mathcal{N}}$ turns out to be a generalized $SU(1,1)$ coherent state of condensed couples of A and $\tilde{A}$ modes (Perolomov 1986; Celeghini et al. 1992), which are entangled modes in the infinite volume limit. The vacuum $|0(\theta)\rangle_{\mathcal{N}}$ is therefore a state densely filled with couples of A and $\tilde{A}$: It is a *coherent condensate* of the couples $A\,\tilde{A}$.[14]

For notational simplicity from now on we will denote by A and $A^\dagger$ the operators S^- and S^+ in (9), respectively. Thus, the doubling process implies that correspondingly we also have $\tilde{S}^-$ and $\tilde{S}^+$, which will be denoted as $\tilde{A}$ and $\tilde{A}^\dagger$, respectively.

One can show that ${}_{\mathcal{N}}\langle 0|0(\theta)\rangle_{\mathcal{N}} \to 0$ and ${}_{\mathcal{N}}\langle 0(\theta')|0(\theta)\rangle_{\mathcal{N}} \to 0$, $\forall \theta \neq \theta'$, in the infinite volume limit $V \to \infty$ (Celeghini et al. 1992, 1998). Thus we conclude that the state space splits in infinitely many physically inequivalent representations in such a limit, each representation labeled by a θ-set $\{\theta_{\mathbf{k}}, \forall \mathbf{k}\}$. This is the *$\theta$-foliation*

[14] In language, in first approximation, the vacuum state is silence. Just like in the present algebraic formalism, there are many kinds of silence. Not only how a silence gap is interpreted in the unfolding of a conversation, but in a more specific and more technical sense. There is, literally, a syntax of silence (Merchant 2001) in linguistic constructions called ellipsis (*Mary bought a book and Bill did ____ too*) and sluicing (*Ann danced with someone but I do not know who ____*). Jason Merchant and other syntacticians and semanticists have persuasively shown that what can be omitted is never just an arbitrary 'bunch of words', but an entire syntactic constituent (an entire Verb Phrase, most frequently). The syntax and semantics of several well defined but unpronounced elements has been part of the theory since the beginning of GG (Chomsky 1955/1985).

process of the state space (already mentioned in subsection 3.1). In the present case of linguistics this represents the process of generation of the manifold of concepts. It is a dynamical process since the generator $G_{\mathbf{k}}$ (see Appendix C for its definition) is essential part of the system Hamiltonian (Celeghini et al. 1992) . Thus in linguistics the 'manifold of concepts' is made of 'distinct', different spaces (the 'physically inequivalent' representations), each one representing a different 'concept' (in language we have the LFs composing the global LF of the entire sentence), here described as the coherent collective mode generated through the X-bar tree as illustrated in subsection 2.1.

These spaces (concepts) are protected against reciprocal interferences since the spaces are 'unitarily inequivalent', i.e. there is no unitary operator able to transform one space in another space (Vitiello 1995, 2001), which corresponds to the fact that syntactic Phases cannot be commingled, nor 'reduced' one into the other. Phases are, as we said above, mutually impenetrable. In practice, however, the unitary inequivalence is smooted out by realistic limitations, such as, for example, the impossibility to reach in a strict mathematical sense the $V \to \infty$ limit (i.e. the 'infinite number' of lexical elements or the theoretically infinite number of choices for the co-referentiality indices in the logical form of even the simplest sentences).[15] Thus, realistically, we may also move from concept to concept in a chain or trajectory going through the manifold of concepts (Vitiello 1995, 2004a, 2004b; Freeman & Vitiello 2006, 2008; Capolupo et al. 2013). These trajectories may be thought as producing 'association of concepts' in their evolving through the manifold of concepts. Remarkably, one may have a multiplicity of such 'associations', each one produced by a specific trajectory, among the many possible ones. One may thus follow different, distinct, non-interfering paths in the space of the concepts. Such features are indeed implied by the fact that the trajectories, although deterministically evolving, are found to be chaotic trajectories (Vitiello 2004b). This corresponds to the compositionality of meanings, when the syntactic derivation proceeds 'upwards' (that is, forward) from the lower Phases to the higher Phases, from local LFs to the composition of more inclusive LFs.

In order to better understand the role played by the 'tilde copies', $\tilde{A}$, it is interesting to compute $N_{A_{\mathbf{k}}} = A^{\dagger}_{\mathbf{k}} A_{\mathbf{k}}$ in the state $|0(\theta)\rangle_{\mathcal{N}}$:

(10) $$\mathcal{N}_{A_{\mathbf{k}}}(\theta) \equiv {}_{\mathcal{N}}\langle 0(\theta)| A^{\dagger}_{\mathbf{k}} A_{\mathbf{k}} |0(\theta)\rangle_{\mathcal{N}} = {}_{\mathcal{N}}\langle 0(\theta)| \tilde{A}_{\mathbf{k}}(\theta) \tilde{A}^{\dagger}_{\mathbf{k}}(\theta) |0(\theta)\rangle_{\mathcal{N}} = \sinh^2 \theta_{\mathbf{k}}$$

From this we see that for any $\mathbf{k}$ the only non-vanishing contribution to the number of non-tilde modes $\mathcal{N}_{A_{\mathbf{k}}}(\theta)$ comes from the tilde operators, which can be expressed by saying that these last ones constitute the dynamic *address* for the non-tilde modes (the reverse is also true, the only non-zero contribution to $\mathcal{N}_{\tilde{A}_{\mathbf{k}}}(\theta)$ comes from the non-tilde operators). In the case of language, this 'address' corresponds to the link between the two copies, or among a chain of cyclic copies in more complex sentences.

In conclusion, the physical content of $|0(\theta)\rangle_{\mathcal{N}}$ is specified by the $\mathcal{N}$-set $\equiv \{\mathcal{N}_{A_{\mathbf{k}}}(\theta), \mathcal{N}_{A_{\mathbf{k}}}(\theta) = \mathcal{N}_{\tilde{A}_{\mathbf{k}}}(\theta), \forall \mathbf{k}\}$, which is called the *order parameter*. It is a characterizing

[15] One of the leaders in the semantics of natural languages wrote (Heim 1983: 232): "We just focused on a particular logical form that grammar provides for the sentence 'She hit it' [...] But there are infinitely many others, since the choice of indices is supposed to be free. So [the simple logical form there reported] represents really only one of many readings that the sentence may be uttered with."

parameter for the vacuum $|0(\theta)\rangle_{\mathcal{N}}$ and explains the meaning of the $\mathcal{N}$ subscript introduced above.

All of this, therefore, sheds some light on the relevance of 'copies' in the MP. In some sense they are crucial in determining (indeed providing the address of) the whole conceptual content of the considered linguistic structure. They provide the *dynamic reference* for the non-tilde modes. Unpronounced copies, being silent, do not reach the SM system, but they are crucially interpreted by the CI system. They are necessary to the understanding of the meaning of what is actually pronounced. Remarkably, they are 'built in' in the scheme here proposed; they are not imposed by hand by use of some constraint 'external' to the linguistic system. It is in this specific sense that we speak of 'self-consistency': Our formal scheme is computationally (logically) self-contained. Perhaps the real power of the linguistic tool available to humans consists in such a specific feature.

4. Concluding Remarks

The essence of the contribution we propose in this paper for the understanding and the physical modeling of the Minimalist Program consists in having pointed out the *dynamical* nature of the transition from a numeration of lexical items to syntax and from syntax to the logical form (LF) of the sentence and from LF to meaning. This has brought us to the identification of the manifold of concepts, to the self-similar properties of the X-bar trees and of their dissipative character (breakdown of the time-reversal symmetry), to the role of the copies in the conceptual intentional system CI. The Hopf algebra structure has shown that the doubled tilde operators, which we have seen to play the role of the copies in the CI system, are 'built in' in the computationally self-contained algebraic scheme. These copies or tilde modes have been recognized to provide the *dynamical reference* (the 'address') of the non-tilde modes. The result is the logical self-consistency (inclusion of the *reference* terms) of languages.

We have also pointed out the mechanism of the foliation of the space of the states, out of which the great richness of the conceptual content, the 'multiplicity' of inequivalent meanings (nested LFs) emerges (see the comments following (10) and the remark by Heim in footnote 15). In this connection, we would like to call the attention of the reader on a further aspect of the scheme we propose in order to model some features of the MP, namely on its intrinsic thermodynamic nature. It is indeed well known (Umezawa 1993) that within the scheme one can consistently define thermodynamic quantities (operators) such as the entropy and the free energy. Let us consider here the entropy.

Thinking of the entropy as an 'index' or a measure of the degree of ordering present in the state of the system (lower entropy corresponding to higher degree of order), one can show that the state $|0(\theta)\rangle_{\mathcal{N}}$ can be constructed by the use of the entropy operator S (Celeghini et al. 1992; Umezawa 1993; Blasone et al. 2011). Its expectation value in $|0(\theta)\rangle_{\mathcal{N}}$ is given by the familiar form, where $W_n \equiv W_n(\theta)$ is some quantity, which here we do not need to specify:

(11) $$_{\mathcal{N}}\langle 0(\theta)|S|0(\theta)\rangle_{\mathcal{N}} = \sum_{n=0}^{+\infty} W_n \log W_n$$

Remarkably and consistently with the breakdown of time-reversal symmetry in dissipative systems (the appearance of the *arrow of time*), time evolution can be shown to be controlled by the entropy variations (De Filippo & Vitiello 1977; Celeghini et al. 1992). These indeed control the variations in the $A - \tilde{A}$ content of $|0(\theta)\rangle_{\mathcal{N}}$, thus controlling the time evolution (the trajectories) in the manifold of concepts (the space of the *infinitely many* LF, see Heim 1983 in footnote 15). Entropy is thus related with the semantic level of the LF, *meanings*, which are dynamically arising as collective modes out of the syntactic (atomistic) level of the basic lexical elements.

In conclusion, we have uncovered the isomorphism between the physics of many-body systems and the linguistic strategy of the Minimalist Program. Although we have exploited the algebraic properties of the many-body formalism, in our scheme the linguistic structures are 'classical' ones. It is known, on the other hand, that the many-body formalism is well suited to describe not only the world of elementary particle physics and condensed matter physics, but also macroscopically behaving systems characterized by ordered patterns (Umezawa 1993; Blasone et al. 2011). Our discussion seems to imply that the crucial mechanism of the foliation of the space of the states has to do with the basic dynamics underlying the linguistic phenomena observed at a macroscopic level. It is an interesting question whether the basic dynamics underlying the richness of the biochemical phenomenology of the brain behavior (Vitiello 1995, 2001; Freeman & Vitiello 2006, 2008; Capolupo et al. 2013; Freeman et al. 2015) also provides the basic mechanisms of linguistics.

Appendix A: On Pauli Matrices and Their Algebra

Consider the 2×2 matrices σ_1, σ_2, σ_3 and the unit matrix I:

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The σ_1, σ_2, σ_3 are the Pauli matrices. They were introduced as an elegant device in the treatment of magnetic spin. The formalism, however, is directly applicable to any system that has two possible states (Perelomov 1986). The space of states on which the matrices operate is built indeed on the basis vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which we will denote by $|0\rangle$ and $|1\rangle$, respectively. The scalar product is denoted by $\langle i|j\rangle = \delta_{ij},\ i, j = 0, 1$. The Pauli matrices satisfy the $su(2)$ algebra, which, in terms of the matrices $\sigma^{\pm} = \sigma_1 \pm i\,\sigma_2$, $\sigma^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, is given by the commutation relations

$$[\sigma_3, \sigma^{\pm}] = \pm\sigma^{\pm}, \quad [\sigma^{-}, \sigma^{+}] = -2\sigma_3 .$$

When we have a collection of N objects ('particles' or 'lexical elements'), which are represented for each i by the 'ground states' $|0\rangle_i$ and 'excited states' $|1\rangle_i$, $i = 1, 2, 3, \ldots N$, we have $\sigma^{\pm} = \sum_{i=1}^{N} \sigma_i^{\pm}$ and $\sigma_3 = \sum_{i=1}^{N} \sigma_{3i}$.

We also write $\sigma_{3i} = \frac{1}{2}(|1\rangle_{ii}\langle 1| - |0\rangle_{ii}\langle 0|)$, with eigenvalues $\pm\frac{1}{2}$, $\sigma_i^{+} = |1\rangle_{ii}\langle 0|$ and $\sigma_i^{-} = |0\rangle_{ii}\langle 1|$.

Appendix B: Dynamical Rearrangement of the $SU(2)$ Symmetry

Consider the state $|l\rangle$ introduced at the beginning of subsection 3.1, namely the state which is a superposition of all states with l elements in $|1\rangle$ and $N-l$ elements in $|0\rangle$. Its explicit form is given by:

$$\begin{aligned} |l\rangle \quad &\equiv \quad [\,|0_1 0_2 \ldots 0_{N-l}\, 1_{N-l+1} 1_{N-l+2} \ldots 1_N\rangle + \ldots \\ &\qquad + |1_1 1_2 \ldots 1_l\, 0_{l+1} 0_{l+2} \ldots 0_N\rangle\,] / \sqrt{\tbinom{N}{l}} \end{aligned}$$

For any l we have (Beige et al. 2005; Blasone et al. 2011):

$$\begin{aligned} \sigma^+ |l\rangle &= \sqrt{l+1}\,\sqrt{N-l}\,|l+1\rangle\,, \\ \sigma^- |l\rangle &= \sqrt{N-(l-1)}\,\sqrt{l}\,|l-1\rangle \end{aligned}$$

This shows that $\sigma^\pm$ and σ_3 acting on $|l\rangle$ may be represented as (the so-called Holstein-Primakoff non-linear realization; Holstein & Primakoff 1940; Shah et al. 1974; De Concini & Vitiello 1976; Blasone et al. 2011)

$$\sigma^+ = \sqrt{N} S^+ A_S\,, \;\; \sigma^- = \sqrt{N} A_S S^-\,, \;\; \sigma_3 = S^+ S^- - \tfrac{1}{2} N,$$

with $A_S = \sqrt{1 - S^+ S^- / N}$, $S^+|l\rangle = \sqrt{l+1}\,|l+1\rangle$ and $S^-|l\rangle = \sqrt{l}\,|l-1\rangle$, for any l. The σ's still satisfy the su(2) algebra (cf. Appendix A). However, in the large N limit, we have:

$$\sigma^\pm\,|l\rangle = \sqrt{N}\,S^\pm\,|l\rangle$$

and thus $S^\pm = \sigma^\pm/\sqrt{N}$ for large N: The phenomenon of the contraction of the algebra occurs (Inönü & Wigner 1953; De Concini & Vitiello 1976; Beige et al. 2005). This means that in the large N limit the su(2) algebra written in the space of the $|l\rangle$ states, for any l, in terms of $S^\pm$ and $S_3 \equiv \sigma_3$, contracts to the so-called (projective) e(2) algebra:

$$[S_3, S^\pm] = \pm S^\pm\,, \quad [S^-, S^+] = 1\,,$$

which is the equation (9) in the subsection 3.1. This is a centra result. It expresses the 'rearrangement' of the su(2) algebra in the e(2) algebra, which is isomorph to the Heisenberg-Weyl algebra (Perelomov 1986), with S_3 playing the role of the number operator and $S^\pm$ the role of ladder operators. The *rearrangement of symmetry* is a well known *dynamical* process (De Concini & Vitiello 1976; Umezawa 1993), which occurs when there is spontaneous breakdown of symmetry characterized by a non-vanishing classical field called order parameter. In the present case, the order parameter is given by $\langle l|\sigma_3|l\rangle = l - \frac{1}{2}N \neq 0$.

Appendix C: Doubling of the Degrees of Freedom for Dissipative Systems

Let us denote by $\mathcal{A}$ the operator algebra of a given system. The algebra mapping $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ defines the doubling of the degrees of freedom of the system. It is a natural requirement to be satisfied when one has to consider, for example, the total energy of a system of two identical particles, $\mathcal{E}_{tot} = \mathcal{E}_1 + \mathcal{E}_2$, or their total angular momentum $L_{tot} = L_1 + L_2$. These sums are defined in the algebra $\mathcal{A} \times \mathcal{A}$ and denote the Hopf coproducts $\mathcal{E}_{tot} = \mathcal{E} \times 1 + 1 \times \mathcal{E}$ and $L_{tot} = L \times 1 + 1 \times L$, respectively, which are commutative under the exchange of the two considered particles.

As said in the subsection 3.2, most interesting is the case of two elements which cannot be treated on the same footing, as it happens when dealing, for example, with open or dissipative systems (e.g. finite temperature systems), where the system elements cannot be exchanged with the elements of the bath or environment in which the system is embedded, or as in the case of linguistics where, at the syntactic and semantic levels, lexical elements, as well as conceptual contents, cannot be simply interchanged. In these cases, we need to consider *q-deformed* Hopf algebras with *noncommutative* Hopf coproducts $\Delta A_q = A \times q + q^{-1} \times A \equiv A\,q\, +\, q^{-1}\,\tilde{A}$ (Celeghini et al. 1998), with the operator (matrix) $A \in \mathcal{A}$ and q a number chosen on the basis of some mathematical constraint on which we do not need to comment here. The doubled operators in the doubling of the algebra $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is denoted by the 'tilde' operators $\tilde{A}$.

For simplicity we are omitting subscripts $\mathbf{k}$ denoting properties of the A (and $\tilde{A}$) modes, e.g. $A_{\mathbf{k}}$, as far as no misunderstanding occurs.

In conclusion, we have the 'copies' $\tilde{A}$ of the operators A, the Hopf doubling of the algebra $A \to \{A,\, \tilde{A}\}$ and of the state space $\mathcal{F} \to \mathcal{F} \times \tilde{\mathcal{F}}$. The operators A and $\tilde{A}$ act on $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively, and commute among themselves.

By using the so-called deformation paramenter $q(\theta)$, with $q(\theta) = e^{\pm\,\theta}$, one obtains (Celeghini et al. 1998; Blasone et al. 2011) the operators $A(\theta)$, $\tilde{A}(\theta)$ and the so-called Bogoliubov transformations:

$$\begin{aligned} A(\theta) &= A \cosh\,\theta - \tilde{A}^\dagger \sinh\,\theta, \\ \tilde{A}(\theta) &= \tilde{A} \cosh\,\theta - A^\dagger \sinh\,\theta. \end{aligned}$$

The canonical commutation relations (CCR) are

$$[A(\theta), A(\theta)^\dagger] = 1, \qquad [\tilde{A}(\theta), \tilde{A}(\theta)^\dagger] = 1.$$

All other commutators equal to zero. The Bogoliubov transformations provide an explicit realization of the doubling or 'copy' process discussed above.

The state annihilated by A and $\tilde{A}$ is denoted by $|0\rangle \equiv |0\rangle \times |0\rangle : A|0\rangle = 0 = \tilde{A}|0\rangle$ (the vacuum state). $A(\theta)$ and $\tilde{A}(\theta)$ do not annihilate $|0\rangle$. They annihilate the state $|0(\theta)\rangle_{\mathcal{N}}$ (Celeghini et al. 1992; Umezawa 1993; Blasone et al. 2011) given by

$$|0(\theta)\rangle_{\mathcal{N}} = e^{i \sum_{\mathbf{k}} \theta_\kappa G_{\mathbf{k}}} |0\rangle = \prod_{\mathbf{k}} \frac{1}{\cosh\,\theta_\kappa} e^{(\tanh\theta_\kappa A^\dagger_\kappa \tilde{A}^\dagger_\kappa)}\,|0\rangle,$$

where θ denotes the set $\{\theta_\kappa, \forall \mathbf{k}\}$. As usual, the symbol $\dagger$ in $A^\dagger$ denotes the hermitian conjugate matrix, namely the transpose and complex conjugate of the matrix representation of A. In the operator $e^{(i \sum_{\mathbf{k}} \theta_\kappa G_{\mathbf{k}})}$, $G_{\mathbf{k}} \equiv -i\,(A^\dagger_{\mathbf{k}} \tilde{A}^\dagger_{\mathbf{k}} \,-\, A_{\mathbf{k}} \tilde{A}_{\mathbf{k}})$. $G_{\mathbf{k}}$ is called the generator of the Bogoliubov transformations and of the state $|0(\theta)\rangle_{\mathcal{N}}$.

We have ${}_{\mathcal{N}}\langle 0|0(\theta)\rangle_{\mathcal{N}} \to 0$ and ${}_{\mathcal{N}}\langle 0(\theta')|0(\theta)\rangle_{\mathcal{N}} \to 0, \forall \theta \neq \theta'$, in the infinite volume limit $V \to \infty$. As already observed in subsection 3.2, this shows that the state space splits in infinitely many physically inequivalent representations in such a limit, each representation labeled by a θ-set $\{\theta_{\mathbf{k}} \,=\, \ln\, q_{\mathbf{k}},\, \forall \mathbf{k}\}$. This is the $q(\theta)$-*foliation* dynamical process of the state space.

Appendix D: Some Useful Formulas on Fibonacci Matrix

The matrix

$$F \equiv \frac{1}{2}\,I + \sigma_3 + 2\,\sigma_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is called the Fibonacci matrix. For the n-powers F^n of the F matrix, with $n \neq 0$, we have

$$F^n \;=\; \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = F_{n-1}\,I + F_n\,F, \qquad \text{n} \neq 0$$

where the matrix elements $F_{n+1},\ F_n,\ F_n,\ F_{n-1}$, with $F_0 \equiv 0$, for any $n \neq 0$, are the numbers in the Fibonacci progression $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13, \ldots$ Moreover, also the coefficients of the matrices I and F in the last member on the r.h.s. of the above relation are the Fibonacci numbers F_{n-1} and F_n. We can indeed verify that

$$\begin{aligned}
F^1 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = F, \\
F^2 &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = I + F, \\
F^3 &= \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = I + 2\,F, \\
F^4 &= \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = 2\,I + 3\,F, \\
F^5 &= \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix} = 3\,I + 5\,F, \\
F^6 &= \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} = 5\,I + 8\,F, \\
\ldots \;\; & etc. \;\; \ldots,
\end{aligned}$$

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